Piecewise isométries

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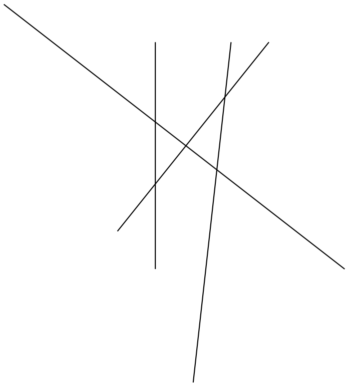
Definitions

Consider a finite number of hyperplanes in $\mathbb{R}^n$, denoted $H_1 \dots H_k$. Let $X = \mathbb{R}^n \setminus \bigcup_{i \leq k} H_i$. A **piecewise isometry** of $\mathbb{R}^n$ is a map T:

Definitions

Consider a finite number of hyperplanes in $\mathbb{R}^n$, denoted $H_1 \dots H_k$. Let $X = \mathbb{R}^n \setminus \bigcup_{i \leq k} H_i$. A **piecewise isometry** of $\mathbb{R}^n$ is a map T:

- defined on X,
- the restriction of T to a connected set is an isometry of $\mathbb{R}^n$,
- the map is one to one (not essential).



- The orbit of a point m is defined for almost every point m.
- The topological entropy is null.

Combinatorics on words.

The **coding** of the map is to associated a letter to each isometry. Define a map ϕ:

$$\phi : X \mapsto \{1 \dots l\}^{\mathbb{N}}$$

$$\phi(m) = (u_n)_{n \in \mathbb{N}}$$

u_n is the name of the isometry defined on a neigborhood of $T^n m$. Define $\Sigma = \overline{\phi(X)}$, as the closure for the product topology.

Questions

$\Sigma =$ language of the map.

- Complexity.
- Combinatorial properties.
- Dynamical properties.

Words and pictures

$$(\Sigma, S)$$

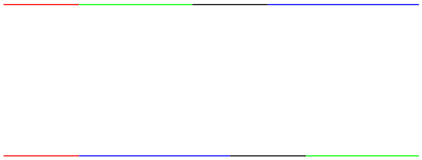
$$(X, T)$$

Word v	Cell
Periodic words	Open sets: polygons, disc
Non periodic words	Fractal set
Substitution	Self similarity

Examples

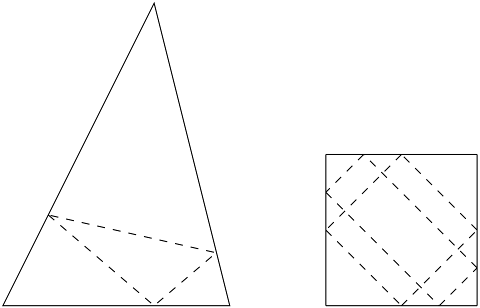
- Interval exchange transformation.
- Interval translation maps.
- Dual billiard
- Billiard in a polytope.
- Polytopes exchange
- Example in $\mathbb{R}^2$.

Interval exchange



$n = 2$, translations.

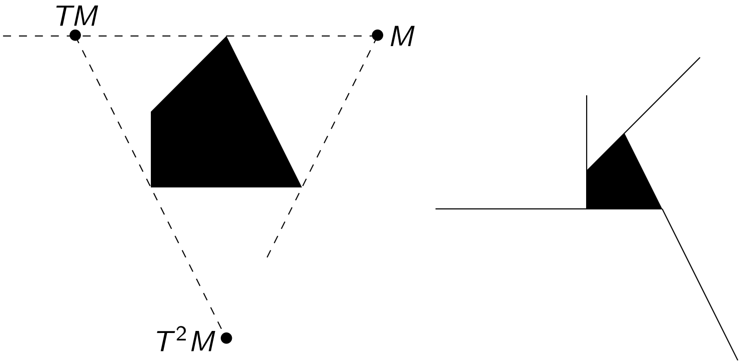
Billiard inside polytope



Billiard inside polygon

$$T : \partial P \times \mathbb{R}^n \to \partial P \times \mathbb{R}^n$$

Dual billiard



$$T : \mathbb{R}^2 \setminus P \to \mathbb{R}^2 \setminus P$$

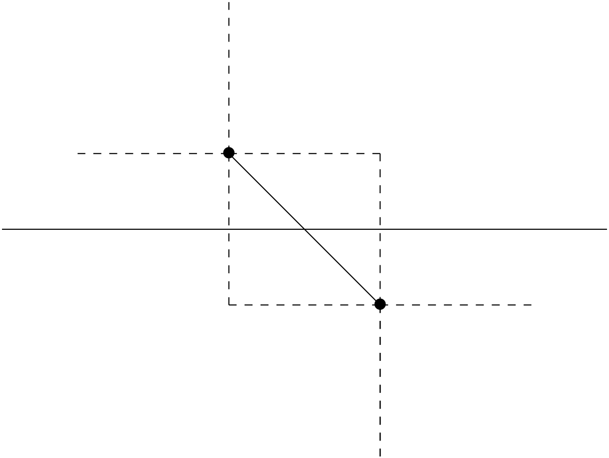
Piecewise rotation: example

$$T : \mathbb{C} \to \mathbb{C}$$

$$T : z \mapsto \begin{cases} e^{i\pi\theta}(z+1) & Im(z) > 0 \\ e^{i\pi\theta}(z-1) & Im(z) < 0 \end{cases}$$

Upper plane: Rotation angle $\pi\theta$
Lower plane: Rotation angle $\pi\theta$
Different centers.

Piecewise rotation



Angle $\pi/2$: periodic points.

History

- Coven, Hedlund, Morse.
- Rauzy, Arnoux
- Boshernitzan
- Cassaigne
- Ferenczi-Zamboni
- Belov-Chernyatev
- Frid
- Smillie-Ulcigrai
- Schwartz, Tabachnikov, Hooper
- Goetz, Quas, Vivaldi

Sturmian words

For $k = 2$:

Sturmian words

Rotation on the circle

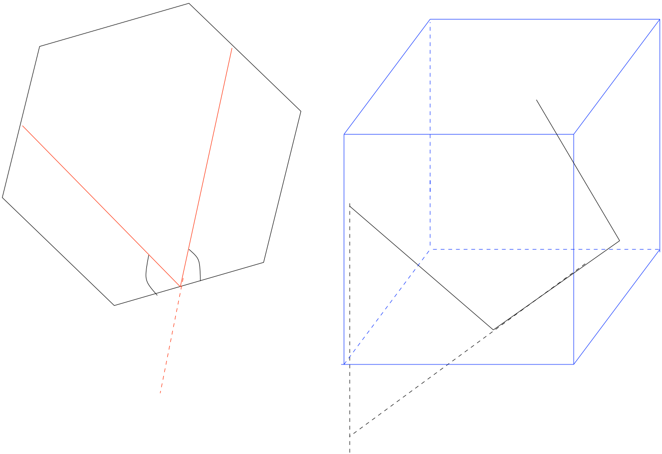
Billiard inside square

Discrete lines in $\mathbb{R}^2$.

Interval exchange, geometry

- Complexity of all sturmian words.
- Complexity of 3 iet.
- Desubstitution.

Billiard



Reflections and billiard.

Coding

Definition

We associate one letter to each face of the polyhedron. In the case of the cube we give the same letters to the parallel faces.

Question Compute $p(n), p(n, q, \omega)$.

Billiard

For a rational polytope:

Billiard flow	Polygon exchange	
Polygon	Interval exchange	
Square	Rotations	Discrete lines

- Complexity
- Language, bispecial words: square, regular octogon.
- Hypercube.
- Periodic orbits.

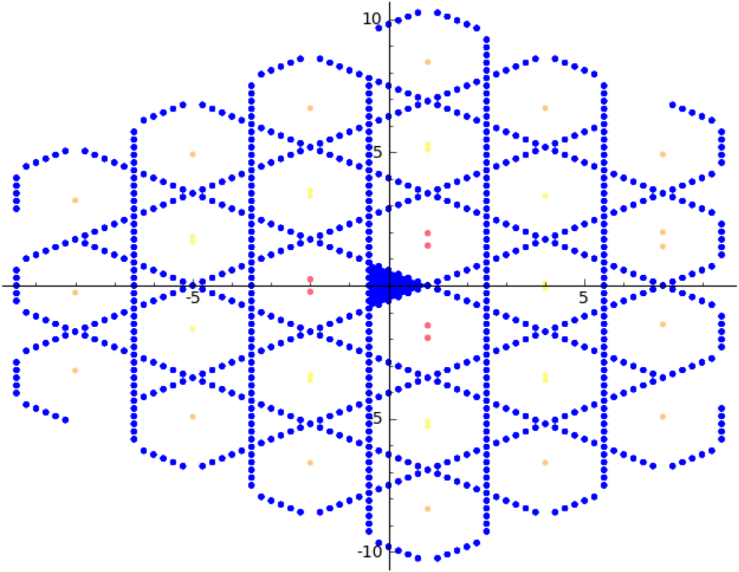
Dual billiard

Some results for

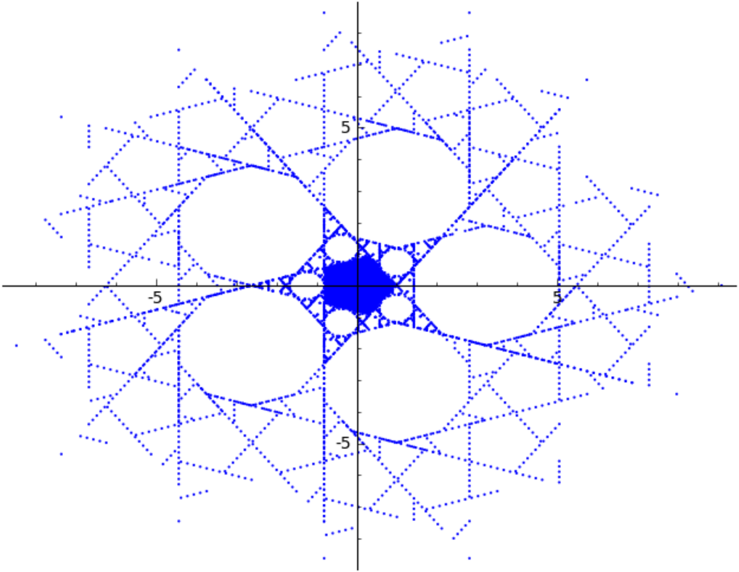
- Regular polygon with $3, 4, 5, 6, 8, 10$ edges.
- Rational polygons.
- Trapezoids.
- Kite.

Complexity, description of language.

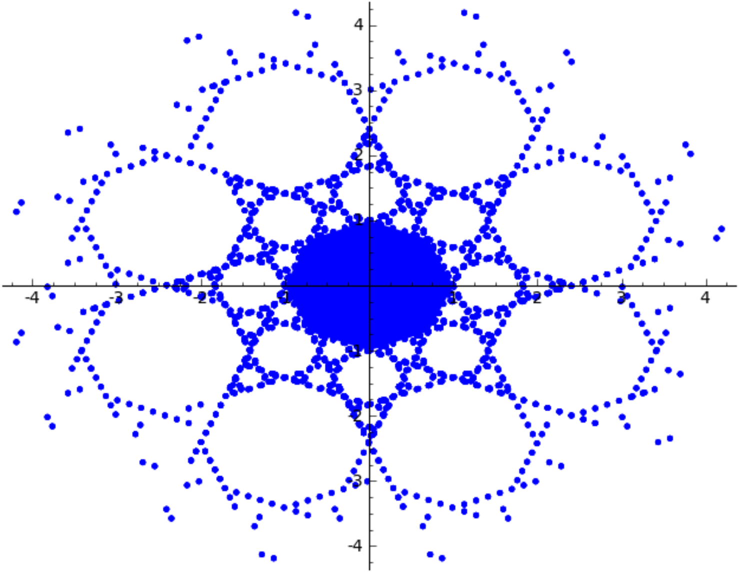
Triangle



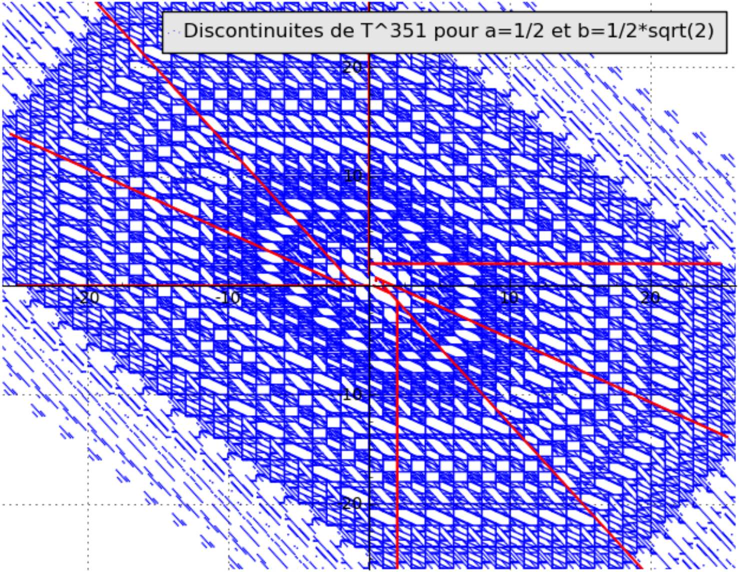
Pentagon



Octogon



Quadrilateral



Questions

- Language ?
- Complexity ?
- Link between maps.

Dual billiard, triangle

- The language is given by

$$\bigcup_{n\in\mathbb{N}}\{1(21)^n, 1(21)^n1(21)^{n+1}\}.$$

- The language is invariant by the substitution $\alpha_{tria} : \begin{cases} 1 \to 121 \\ 2 \to 1^{-1} \end{cases}$.

Dual billiard, regular pentagon

$$\sigma : \begin{cases} 1 \to 1121211 \\ 2 \to 111 \\ 3 \to 3 \end{cases} \qquad \psi : \begin{cases} 1 \to 2232232 \\ 2 \to 232 \\ 3 \to 2^{-1} \end{cases} \qquad \xi : \begin{cases} 1 \to 23222 \\ 2 \to 2 \\ 3 \to 3 \end{cases}$$

If P is the regular pentagon then Z is the union of

$$\bigcup_{n \in \mathbb{N}} \{\sigma^n(1), \sigma^n(12)\},$$

$$\bigcup_{n,m \in \mathbb{N}} \{\psi^m(2), \psi^m(2223), \psi^m \circ \sigma^n(1), \psi^m \circ \sigma^n(12)\},$$

$$\bigcup_{n,m \in \mathbb{N}} \{\psi^m \circ \xi \circ \sigma^n(1), \psi^m \circ \xi \circ \sigma^n(12)\}.$$

- *Periodic orbits= periodic words*
- *Self similarity= fixed point of substitution.*

Two rotations of same angle θ on the plane.

Theorem

- *T is not injective: Globally attracting map.*
- *T is not surjective: Globally repulsing map.*

Theorem

- *T is bijective: Bounded orbits for rational θ.*
- *irrational θ: For every set A of positive measure: almost very point of A visits A infinitely often.*
- *Irrational θ: Lower bound on density of periodic islands:* $3\log 2 - \frac{\pi^2}{8}$.

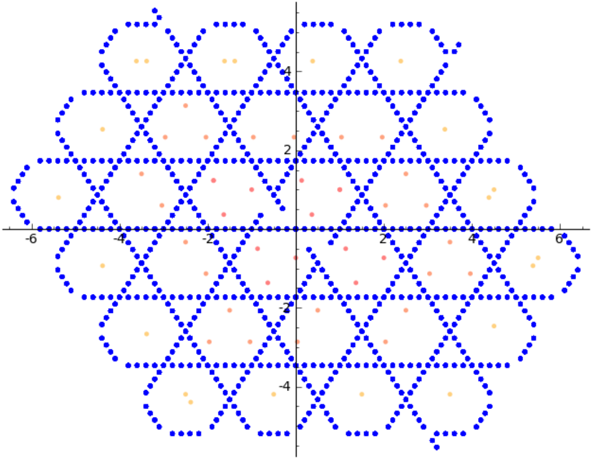


Figure: Angle 1/3

Piecewise isometries

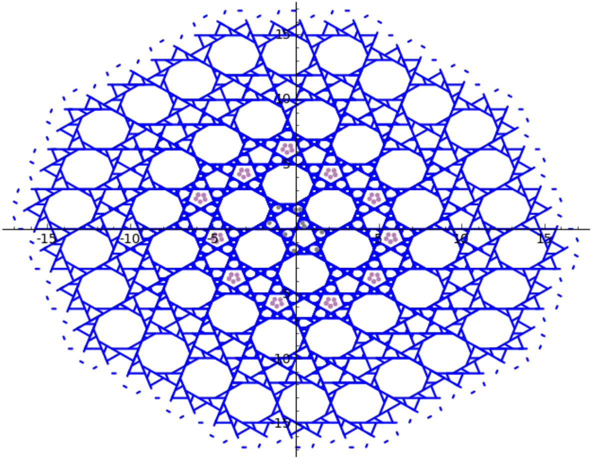


Figure: Angle 2/5

Piecewise isometries

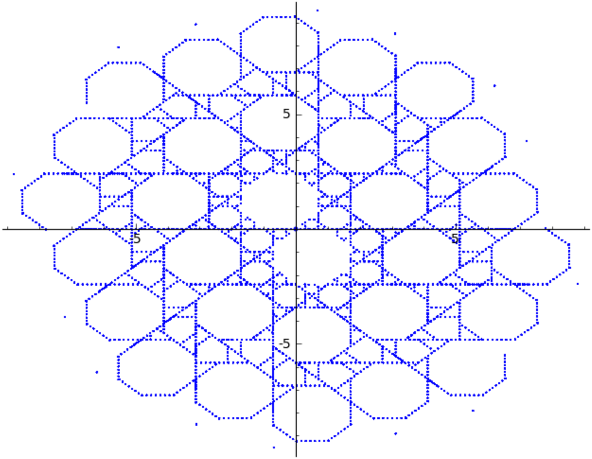


Figure: Angle 1/4

Link with rotations

Consider a rotation of angle θ coded by two letters. For a piecewise rotation of angle θ:

- Sequence $\lim_k \frac{p_k}{q_k} = \theta$
- Periodic words of Rotation of angle $\frac{p_k}{q_k}$.
- Periodic words of piecewise rotation.

Piecewise isometries

Angle $\theta = 1/5$. List of words:

- $0^4 1^4$
- $0^3 1^3 0^3 1^2 0^3 1^3 0^2 1^3 0^3 1^3 0^2 1^3$ two pentagons
- $0^3 1^3 0^3 1^2$ big pentagon
- $0^2 1^3 0^3 1^3$ big pentagon
- $0^3 1^3$ big decagon
- $0^3 1^3 0^3 1^2 0^2 1^3 0^2 1^3 0^3 1^3 0^2 1^3$ two pentagons

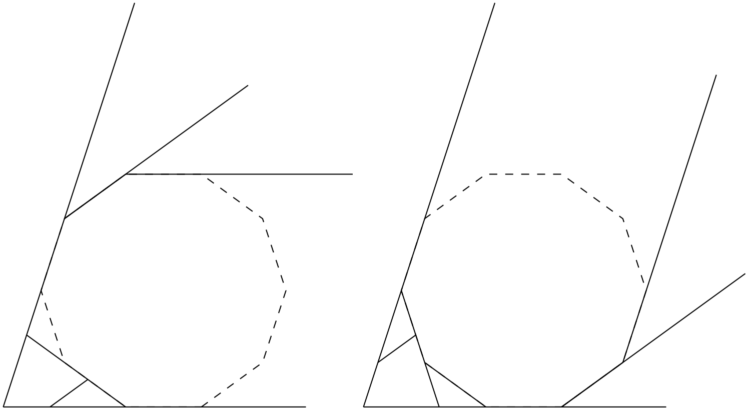
Result

Theorem (B. Kabore)

Consider a piecewise rotation of angle $\pi\theta$.

- *If $\theta \in \{1/2, 1/3, 1/5\}$ complete description of Σ.*
- *Link with dual billiard outside regular polygon.*

Return map



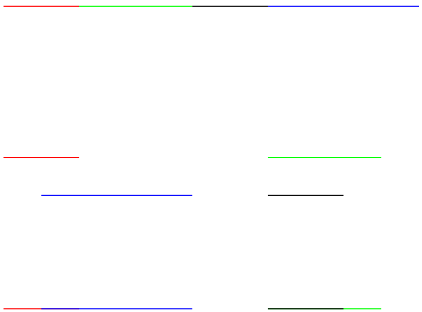
Questions

Consider a piecewise isometry:

Conditions to enforce the language to contain fixed point of substitution.

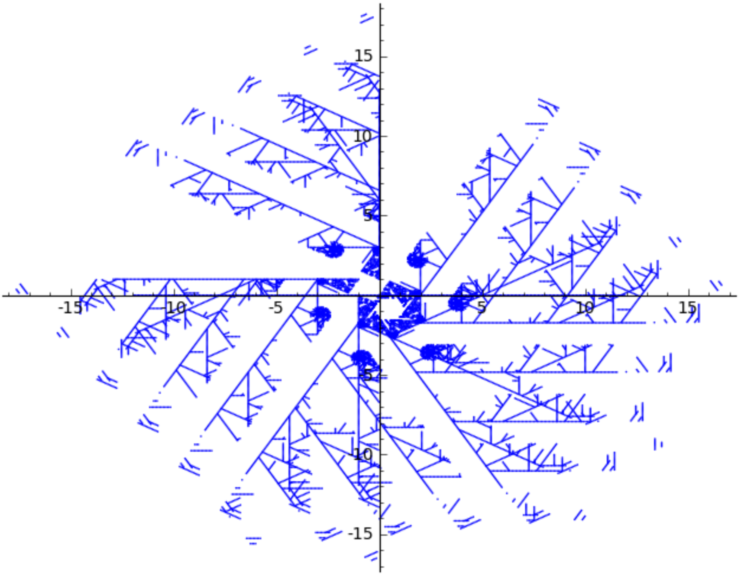
Complexity of a general piecewise isometry.

Interval translations



Non bijective map.

Piecewise isometries



- Alphabet $\{0 \ldots d-1\}$.
- Infinite word $u \in \mathcal{A}^{\mathbb{N}}$.
- Factor $u_i \ldots u_{i+n-1}$ of u.
- Language: union of factors of u.

Complexity

Definition

If v is an infinite word, we define the COMPLEXITY function $p(n, v)$ as the number of different words of length n inside v.

Example

$v = 0100011011000\ldots \quad p(n, v) = 2^n$

Complexity

Definition

If v is an infinite word, we define the COMPLEXITY function $p(n, v)$ as the number of different words of length n inside v.

Example

$v = 01011111\ldots \;\; p(n, v) = 4, \;\; n \geq 3$

In fact we can compute two different complexities:

- The complexity of one word: $p(n, \phi(m))$.
- The global complexity $p(n)$: Complexity of the language

$$\bigcup_{m \in X} F(\phi(m)).$$

Examples

- Billiard.
- Dual Billard.
- Substitution.
- Interval exchanges.

Billiard

Let P be a polyhedron of $\mathbb{R}^d$, $m \in \partial P$ and $\omega \in \mathbb{RP}^{d-1}$.
The point moves along a straight line until it reaches the boundary of P.
On the face: orthogonal reflection of the line over the plane of the face.

$$X = \partial P \times \mathbb{RP}^{d-1}$$

$$T : \quad X \longrightarrow X$$

$$T : \quad (q, \omega) \mapsto (q', \omega').$$

If a trajectory hits an edge, it stops.

Definition

- P is rational if the angles of P are in $\pi\mathbb{Q}$.
- P is rational if the vertices of P are on a lattice of $\mathbb{R}^2$.

Substitution

A substitution is a morphism of free monoid. For example for $\{0;1\}^*$ we have:

$$\phi \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$$

$$\phi^2(0) = 010, \phi^3(0) = 01001.$$

$$v = \lim_{n \to +\infty} \phi^n(0), v = \phi(v).$$

$$v = 0100101001001010\ldots$$

For a fixed point v of a substitution, the dynamical system is (X, S) where

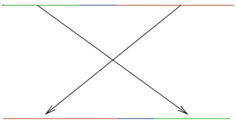
$$X = \overline{\bigcup_{n \in \mathbb{N}} S^n v}.$$

Question Compute $p(n, v)$.

Interval exchange

$$X = [0;1)$$

T is locally a translation, and is a bijective map.



Interval exchange

Question Compute $p(n, v), p(n)$. $p(n)$ is the complexity of all the words obtained as a coding of an interval exchange with d intervals.

		Billiard		Dual	
	rational	General	Cube $\mathbb{R}^{d+1}$	rational	Regular
$p(n,u)$	n	$h_{top}=0$	n^d	1	n
$p(n)$	n^3	$h_{top}=0$	n^{3d}	n^2	n^2

	Ei 2	Ei 3	Subst
$p(n,u)$	n	n	$1, n, n\log\log n, n\log n, n^2$
$p(n)$	n^3	n^4	

Bispecial words

Let $\mathcal{L}(n)$ the set of words of length n in a language. For $v \in \mathcal{L}(n)$ let

$$\begin{aligned}
s(n) &= p(n+1) - p(n).\\
m_l(v) &= card\{a \in \Sigma, \quad av \in \mathcal{L}(n+1)\}.\\
m_r(v) &= card\{b \in \Sigma, \quad vb \in \mathcal{L}(n+1)\}.\\
m_b(v) &= card\{(a,b) \in \Sigma^2, \quad avb \in \mathcal{L}(n+2)\}.\\
b(n) &= \sum_{v \in \mathcal{L}(n)} (m_b(v) - m_r(v) - m_l(v) + 1).
\end{aligned}$$

Definition

A word v is:

- right special if $m_r(v) \geq 2$,
- left special if $m_l(v) \geq 2$,
- bispecial if it is right and left special.

We have

Lemma (Cassaigne 97)

For all integer n we have

$$s(n+1) - s(n) = b(n).$$